Adding Optionality

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Introduction

- In 1986, Robert Fulghum released a book called "All I Really Need to Know I Learned in Kindergarten". The book was a NYTimes best seller.
- My mother taught me to add positive numbers when I was in kindergarten.
- A UCLA finance professor named Herb Johnson taught me how to price options twenty years later.
- The only point this talk will make is that one can treat the valuation of optionality as akin to adding positive numbers.
- We will illustrate the principle by valuing vanilla and exotic financial products using nothing more than the Pythagorean theorem.

The Addition Group and 1 Dimensional Lie Groups

- In abstract algebra, a group is a set equipped with a binary operation that combines any 2 elements to form a 3rd element in such a way that 4 conditions called group axioms are satisfied, namely closure, associativity, identity and invertibility.
- An example is the additive group $(\mathbb{R}; +, 0)$ taught in elementary school.
- Wiki defines a 1-parameter group as a continuous group homomorphism $\varphi : \mathbb{R} \to G$ from the real line $\mathbb{R}$ (as an additive group) to some other topological group G. In particular, every 1-dimensional Lie group is locally isomorphic to $\mathbb{R}$, see Lie Groups, P. M. Cohn 1957, page 58.
- 2 days ago, a wise mathematician named John Moody wrote me that

  "Any two one-parameter groups are locally isomorphic"

  is a true, meaningful and deep fact of mathematics which separates student level Maths from research level Maths.

The Addition Monoid and Optionality

- If invertibility is dropped as a group axiom, then a group generalizes to a monoid. The 3 monoid axioms are closure, associativity, and identity.
- For example, $([0,\infty);+,0)$ is the additive monoid. Roughly speaking, you can add, but you can't necessarily subtract.
- An isomorphic monoid arises from $([0,\infty);+,0)$ through an invertible function $G:[0,\infty)\mapsto S$ called a generator. In the monoid $(S;\oplus,G(0))$, $\oplus$ is: $g_1 \oplus g_2 \equiv G(G^{-1}(g_1)+G^{-1}(g_2))$ for $g_1 \in S, g_2 \in S$.
- For example if the generator $G(a)=e^a, a \geq 0$ is the exponential function, then $S=[1,\infty)$, $G(0)=1$ is the identity, and $g_1 \oplus g_2 \equiv e^{\ln(g_1)+\ln(g_2)} = g_1 \times g_2$. In words, ordinary multiplication of numbers ≥ 1 is isomorphic to ordinary addition of non-negative numbers.
- A married put written on an asset with future spot price $S_T \geq 0$ pays off $S_T \vee K$ for $K \geq 0$ when it matures at T. In a zero rates world, we value a married put as $S_0 \oplus K$, where S_0 is the current price of the underlying asset. The generator G determining $\oplus$ is given on the next slide.

An Optionality Monoid

- All that we need to create a monoid isomorphic to the additive monoid $([0,\infty);+,0)$ is to pick an invertible generator $G:[0,\infty)\mapsto S$.
- Instead of $G(a)=e^a, a\geq 0$, consider the increasing power function $G(a)=a^b, a\geq 0$ for $b\in(0,1]$. Then $S=[0,\infty)$, $G(0)=0$ is the identity, and the binary operation $g_1\oplus^b g_2\equiv(g_1^{\frac{1}{b}}+g_2^{\frac{1}{b}})^b$ is an $\ell^{\frac{1}{b}}$ norm of $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$.
- Recall that $b\in(0,1]$. When $b=1$, then $g_1\oplus^1 g_2=g_1+g_2$. At the other extreme, $\lim_{b\downarrow 0} g_1\oplus^b g_2=g_1\vee g_2$. $\oplus^b$ interpolates addition $+$ and option $\vee$.
- We interpret b as a bounded dispersion measure of S_T, given S_0. For example, if σ is your favorite dispersion measure, set $b=(1+\frac{1}{\sigma^2 T})^{-1}$.
- We value a married put paying $S_T\vee K$ at T by $S_0\oplus^b K\equiv(S_0^{\frac{1}{b}}+K^{\frac{1}{b}})^b, b\in(0,1]$, which is isomorphic to S_0+K.
- We can also value other options as we now illustrate.

A Warmup on Pricing Linear Payoffs

- Suppose that an underlying asset has known initial price $S_0 > 0$, but unknown future prices $S_1 > 0$ and $S_2 > 0$ at the end of periods 1 and 2.
- We assume no dividends, no frictions, and no arbitrage.
- Suppose that a derivative security pays the gross return $\frac{S_1}{S_0}$ at the end of period 1. Its value is \$1.
- Suppose that a different derivative security pays the product of the gross returns $\frac{S_1}{S_0} \times \frac{S_2}{S_1}$ at the end of period 2. Its value is also \$1.
- Suppose that a third derivative security pays the product of the gross returns $\frac{S_1}{S_0} \times \frac{S_2}{S_1}$ times 3 at the end of period 2. Its value is \$3.
- We obtained a unique value without specifying dynamics because the 3 payoffs are all linear in S_T. On the next slide, we add optionality to the payoffs, so we will need to specify dynamics in order to get a unique value.

Adding Optionality

- Recall that a derivative security paying the product of the gross returns $\frac{S_1}{S_0} \times \frac{S_2}{S_1} \times 3$ at the end of period 2 was valued at \$3.
- We can interpret this payoff as the product of 3 gross returns, where the last gross return $\frac{S_3}{S_2} = 3 = 300\%$.
- Similarly, a claim paying $\frac{S_1}{S_0} \times 4$ at the end of the 2nd period pays off the product of 2 gross returns, where the last gross return is $\frac{S_2}{S_1} = 4 = 400\%$. It is valued at \$4.
- Suppose that we add optionality to both payoffs. One can have either $\frac{S_1}{S_0} \times \frac{S_2}{S_1} \times 3$ or $\frac{S_1}{S_0} \times 4$. The choice is to be made at the end of the 2nd period. Clearly, the value exceeds the larger of 3 and 4, which is 4.
- To get a unique value for this option, we need to specify the risk-neutral dynamics of its underlying asset. On the next slide, we specify risk-neutral dynamics which values this option at 5. The actual valuation formula that we use is $5 = \sqrt{3^2 + 4^2}$.

Multiplicative Random Walk w. Dagum Distributed Ratios

- We assume that the two single period price relatives $\frac{S_1}{S_0}$ and $\frac{S_2}{S_1}$ are IID. The common distribution is called conjugate power Dagum (CPD). The CDF of a random gross return G is: $\mathbb{Q}\{G \leq g\} = \left(1 + g^{-\frac{1}{b}}\right)^{b-1}, b \in (0,1)$.
- The above CDF sets the mean of the gross return at 1. Under $\mathbb{P}$, we can have a different mean gross return $\mu > 0$ in which case:

$$\mathbb{P}\{G \leq g\} = \left(1 + \left(\frac{g}{\mu}\right)^{-\frac{1}{b}}\right)^{b-1}, \qquad b \in (0,1).$$

- This two parameter Dagum model is a heavier tailed alternative to the lognormal model in wide use. Besides being more realistic, it is also more tractable. The quantile function is explicit and option pricing formulas are elementary functions. Furthermore, the married put value is a pseudo-sum of spot price and strike price, i.e. isomorphic to their addition.
- Carr and Torricelli (2020) show that there exist multiple supporting continuous-time stochastic processes whose marginals are all CPD.

Know your Last Return

- The risk-neutral conjugate power Dagum model has a free parameter $b \in (0,1)$ which is a bounded measure of the dispersion of each single period gross return.
- In this model, a single-period married put paying $S_1 \vee K$ at the end of the first period is priced at $S_0 \oplus^b K \equiv (S_0^{\frac{1}{b}} + K^{\frac{1}{b}})^b, b \in (0,1]$. For example if $b = \frac{1}{2}$, $S_0 = 3$, and $K = 4$, then the married put is priced at $5 = \sqrt{3^2 + 4^2}$.
- To take an apparently different example, a claim paying $\frac{S_1}{S_0} \times \frac{S_2}{S_1} \times 3$ or $\frac{S_1}{S_0} \times 4$ at the end of the 2nd period is priced at $5 = \sqrt{3^2 + 4^2}$ when $b = \frac{1}{2}$.
- We add optionality to this payoff on the next slide.

Adding Optionality

- We add optionality to the fixed final return claim paying $\frac{S_1}{S_0} \times \frac{S_2}{S_1} \times 3$ or $\frac{S_1}{S_0} \times 4$ at the end of the 2nd period. Suppose that one receives \$12 for exercising early at time 1. If the claim is exercised early, then 12 can be interpreted as the last gross return, as specified in the claim's term sheet.
- Summarizing the payoff of our Bermudan claim, one can have either the product of 3 gross returns with the last one being 3, the product of 2 gross returns with the last one being 4, or just a single gross return of 12.
- This Bermudan claim is worth at least 12 since exercise is not required at time 1. Not exercising early still leaves the choice at time 2 between a product of 2 returns and a product of 3 returns, with final returns fixed in both cases. The Bermudan claim embeds an option on an option.
- In the lognormal model, valuation involves bivariate normal CDF's.
- In contrast, when $\frac{S_1}{S_0}$ and $\frac{S_2}{S_1}$ are IID conjugate power Dagum with mean 1 and bounded dispersion $b \in (0, 1)$, then valuation is elementary.
- When $b = \frac{1}{2}$, the claim is valued at $13 = \sqrt{12^2 + 5^2} = \sqrt{12^2 + 4^2 + 3^2}$.

Summary

- We proposed an approach in which valuing optionality between two alternatives is isomorphic to adding them.
- We illustrated the approach when the generator is a power function, but a wide variety of generators and corresponding arithmetics can be used.
- Ordinary multiplication $\times$ distributes over both ordinary addition $+$ and the optionality operation $\oplus^b$ arising when the generator is a power function. It can be shown that our optionality operation $\oplus^b$ is the only binary operation between non-negative reals with this property.
- Associativity makes the approach especially powerful for compound options as illustrated by our Bermudan crash put.
- Treating optionality as isomorphic to addition is also useful when the domain of the underlying security's price differs from $(0, \infty)$. For example, one can easily valued (compound) options on underlyings whose values reside eg. in $\mathbb{R}$, in $(0, 1)$, or in $(-1, 1)$. We have used the respective generators $\ln g, g > 0$, $(1 + e^{-a})^{-1}, a \in \mathbb{R}$, and $\tanh(a), a \in \mathbb{R}$ for these 3 domains.