Semi-closed form prices of barrier options in the time-dependent CEV and CIR models

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joined work with Peter Carr and Dmitry Muravey

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The time-dependent constant elasticity of variance (CEV) model is a one-dimensional diffusion process that solves a stochastic differential equation

$$dS_t = \mu(t) S_t dt + \sigma(t) S_t^{\beta+1} dW_t, \qquad S_{t=0} = S_0. \tag{1}$$

Here $t \geq 0$ is the time, S_t is the stochastic stock price, $\mu(t)$ is the drift, $\sigma(t)$ is the volatility and β is the elasticity parameter such that $\beta < 1,\ \beta \neq \{0, -1\}$[1], W_t is the standard Brownian motion.

[1] In case $\beta = 0$ this model is the Black-Scholes model, while for $\beta = -1$ this is the Bachelier, or time-dependent Ornstein-Uhlenbeck (OU) model.

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Let us consider an Up-and-Out barrier Call option $C(t, S)$ written on the underlying process S_t. By the Feynman–Kac formula this price solves the following partial differential equation (PDE)

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2(t) S^{2\beta+2} \frac{\partial^2 C}{\partial S^2} + [r(t) - q(t)] S \frac{\partial C}{\partial S} = r(t) C. \tag{2}$$

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This equation should be solved subject to the terminal condition at the option maturity $t = T$

$$C(T, S) = (S - K)^+, \tag{3}$$

where K is the option strike, and the boundary conditions

$$C(t, 0) = 0, \qquad C(t, H(t)) = 0, \tag{4}$$

where $H(t)$ is the upper barrier, perhaps time-dependent.

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The PDE in Eq.(2) can be transformed to

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial z^2} + \frac{b}{z}\frac{\partial u}{\partial z}, \tag{5}$$

where b is some constant, $u = u(\tau, z)$ is the new dependent variable, and (τ, z) are the new independent variables. The Eq.(5) is the PDE associated with the one-dimensional Bessel process, [Revuz and Yor 1999]

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As far as the terminal (now the initial) condition in Eq.(3) and the boundary conditions in Eq.(4) in the new variables is concerned, we must distinguish two cases, which are determined by the sign of β.

- If $-1 < \beta < 0$, the domain of definition for z is $z \in [0, y(\tau)]$, where $y(\tau)$ can be explicitly represented in terms of the parameters $\mu(t)$, $\sigma(t)$, β and the barrier $H(t)$.

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- Therefore, the price of the Up-and-Out barrier Call option solves the following initial-boundary problem in the curvilinear strip $[0, y(\tau))$:

$$\begin{aligned} \frac{\partial u(\tau,z)}{\partial \tau} &= \frac{1}{2}\frac{\partial^2 u(\tau,z)}{\partial z^2} + \frac{b}{u}\frac{\partial u(\tau,z)}{\partial z}, \qquad (7) \\ u(0,z) &= f(z), \qquad 0 < z < y(0), \\ u(\tau,0) &= 0, \qquad u(\tau, y(\tau)) = 0. \end{aligned}$$

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The time-dependent version of the CIR model can be defined as a solution of the following SDE:

$$dr_t = \kappa(t)[\theta(t) - r_t]dt + \sigma(t)\sqrt{r_t}dW_t, \qquad r_{t=0} = r. \tag{9}$$

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$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2(t) r \frac{\partial^2 C}{\partial r^2} + \kappa(t)[\theta(t) - r]\frac{\partial C}{\partial r} = rC. \tag{10}$$

The terminal condition at the option maturity $T \le S$ for this PDE reads

$$C(T, r) = (F(r, T, S) - K)^+ , \tag{11}$$

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The second boundary can be naturally set at $r \to \infty$. As at $r \to \infty$ the ZCB price tends to zero, the Call option price also vanishes in this limit.

Proposition

The Eq.(10) can be transformed to

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial z^2} + \frac{b}{z}\frac{\partial u}{\partial z}, \tag{13}$$

where b *is some constant,* $u = u(\tau, z)$ *is the new dependent variable, and* (τ, z) *are the new independent variables, if*

$$\frac{\kappa(t)\theta(t)}{\sigma^2(t)} = \frac{m}{2}, \tag{14}$$

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$$u(\tau, z) = q(\tau, z) + \int_{y(0)}^{\infty} u(0, \zeta) q_\tau(z, \zeta, b) d\zeta. \tag{15}$$

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$$q_\tau(z, \zeta, b) = \frac{\sqrt{z\zeta}}{\tau} \left(\frac{\zeta}{z}\right)^b e^{-\frac{z^2+\zeta^2}{2\tau}} I_{b-1/2}\left(\frac{z\zeta}{\tau}\right). \tag{16}$$

Here $I_\nu(x)$ is the modified Bessel function of the first kind, [Abramowitz and Stegun 1964] .

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Therefore, following the general idea of the method of heat potentials, we represent the solution in the form of a generalized potential for the Bessel PDE

$$q(\tau, z) = \int_0^\tau \Psi(k) \frac{\partial}{\partial \xi}\left[\frac{\sqrt{z\xi}}{\tau - k}\left(\frac{\xi}{z}\right)^b e^{-\frac{z^2+\xi^2}{2(\tau-k)}} I_{b-1/2}\left(\frac{z\xi}{\tau - k}\right)\right]\Bigg|_{\xi\to y(k)} dk, \tag{18}$$

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It can be seen that $q(\tau, z)$ solves Eq.(13) as the derivative of the integral on the upper limit is proportional to the Delta function which vanishes due to $z \neq y(\tau)$. The solution in Eq.(18) also satisfies the initial condition at $\tau = 0$, and the vanishing condition at $z \to \infty$.

At the barrier $z = y(\tau)$ function $q(\tau, z)$ is discontinuous. Following a similar approach for the heat potentials method, [Tikhonov and Samarskii 1963]), it can be shown that the limiting value of $q(\tau, z)$ at $z = y(\tau) + 0$ is equal to $\varsigma(\tau)$:

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The Eq.(19) is a linear Volterra equations of the second kind, [Polyanin and Manzhirov 2008] . Since $\varsigma(\tau)$ is a continuously differentiable function, Eq.(19) has a unique continuous solution for $\Psi(\tau)$. The Volterra equation can be efficiently solved either numerically or semi-analytically, see [Itkin and Muravey 2020] for the discussion on various approaches to solving this type of equations.

At the barrier $z = y(\tau)$ function $q(\tau, z)$ is discontinuous. Following a similar approach for the heat potentials method, [Tikhonov and Samarskii 1963]), it can be shown that the limiting value of $q(\tau, z)$ at $z = y(\tau) + 0$ is equal to $\varsigma(\tau)$:

$$\begin{aligned} \varsigma(\tau) = \Psi(\tau) + \\ + \int_0^\tau \Psi(k) \frac{\partial}{\partial y(k)} \left[\frac{\sqrt{y(\tau) y(k)}}{\tau - k} \left(\frac{y(k)}{y(\tau)} \right)^b e^{-\frac{y^2(\tau) + y^2(k)}{2(\tau - k)}} I_{b-1/2} \left(\frac{y(\tau) y(k)}{\tau - k} \right) \right] dk. \end{aligned} \tag{19}$$

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Once Eq.(19) is solved and the function $\Psi(\tau)$ is found, the final solution reads

$$\begin{aligned} u(\tau, z) = \int_0^\tau \Psi(k) \frac{\partial}{\partial y(k)} \left[\frac{\sqrt{z y(k)}}{\tau - k} \left(\frac{y(k)}{z} \right)^b e^{-\frac{z^2 + y^2(k)}{2(\tau - k)}} I_{b-1/2} \left(\frac{z y(k)}{\tau - k} \right) \right] dk \\ + \int_{y(0)}^\infty u(0, \zeta) q_\tau(z, \zeta, b) d\zeta. \end{aligned} \tag{20}$$

Consider now the problem in the domain $[0, y(\tau)]$. The construction of the solution is similar to the presented above. Again, in order to obtain a PDE with a homogeneous initial condition, we represent the solution in the form

$$u(\tau, z) = q(\tau, z) - \varsigma_0(\tau) + \int_0^{y(0)} u(0, \xi) q_\tau(z, \zeta, b) d\zeta, \tag{21}$$

$$\varsigma_0(\tau) = -\int_0^{y(0)} u(0, \zeta) q_\tau(0, \zeta, b) d\zeta,$$

$$q_\tau(0, \zeta, b) = \frac{2^{1/2-b}\zeta^{2b}}{\tau^{b+1/2}\Gamma\left(b+\frac{1}{2}\right)} e^{-\frac{\zeta^2}{2\tau}},$$

where ($\Gamma(x)$ is the Euler Gamma function, [Abramowitz and Stegun 1964]). The method to construct $q(\tau, z)$ is fully analogical to the previous section.

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The double barrier options can also be priced via the BP method. This problem corresponds to the domain $[y(\tau), h(\tau)]$. For doing so, we have to use the so-called double layer potentials:

$$u(\tau, z) = q_y(\tau, z) + q_h(\tau, z) + \int_{y(0)}^{h(0)} u(0, \xi) q_\tau(z, \zeta, b) d\zeta. \tag{22}$$

However, for this case there are two unknown potential density functions $\Psi(\tau)$ and $\Phi(\tau)$ defined as a solution to a system of Volterra equation. For more details, see [Carr, Itkin, and Muravey 2020] .

In this section we describe the method of Generalized integral transform(GIT). We consider the problem in the curvilinear strip $[0, y(\tau)]$. This problem emerges when we consider the CEV process with $\beta < 0$. Since the Laplace transform of Eq.(13) with time-homogeneous coefficients gives rise to the Bessel ODE, [Abramowitz and Stegun 1964] , it would be natural seeking for the general integral transform in the class of Bessel functions.

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$$\bar{u}(\tau, p) = \int_0^{y(\tau)} z^{\nu+1} u(\tau, z) J_{|\nu|}(zp) dz, \tag{23}$$

where $p = a + \mathrm{i}\omega$ is a complex number, $J_\nu(x)$ is the Bessel function of the first kind, and $\nu = 1/(2\beta) < 0$, since $\beta < 0$.

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Next, let us multiply both parts of Eq.(13) by $z^{\nu+1} J_{|\nu|}(zp)$ and integrate on z from 0 to $y(\tau)$. In the result we obtain the following Cauchy problem for $\bar{u}$ (the unknown function $\Psi(\tau)$ should also be found):

$$\frac{d\bar{u}(\tau, p)}{d\tau} = \frac{1}{2}\left[-p^2 \bar{u}(\tau, p) + y^{\nu+1}(\tau) J_{|\nu|}(y(\tau)p)\Psi(\tau)\right], \tag{24}$$

$$\bar{u}(p, 0) = \int_0^{y(0)} z^{\nu+1} J_{|\nu|}\left(zp\right) u(0, z) dz, \quad \Psi(\tau) = \left.\frac{\partial u}{\partial z}\right|_{z=y(\tau)}.$$

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The solution of this problem reads

$$\bar{u} = e^{-p^2\tau/2}\left[\bar{u}(0, p) + \frac{1}{2}\int_0^\tau e^{p^2 s/2}\Psi(s) y^{\nu+1}(s) J_{|\nu|}(y(s)p) ds\right]. \tag{25}$$

Note that if $y(\tau) \equiv 0$ the integral transform Eq.(23) is the well known Hankel transform [Bateman and Erdélyi 1953] :

$$u(\tau, z) = z^{-\nu} \int_0^\infty p J_{|\nu|}(zp) \left[\int_0^\infty \zeta^{\nu+1} J_{|\nu|}(\zeta p) u(\tau, \zeta) dz \right] dp \tag{26}$$

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Once these issues are resolved, we obtain a semi-analytical solution to the problem of pricing Up-and-Out barrier Call options for the CEV model with $\beta < 0$.

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- To build the inverse transform we are looking for the function $u(\tau, z)$ in the form :

$$u(\tau, z) = z^{-\nu} \sum_{n=1}^{\infty} \alpha_n(\tau) J_{|\nu|}\left(\frac{\mu_n z}{y(\tau)}\right). \tag{27}$$

Here μ_n is an ordered sequence of the positive zeros of the Bessel function $J_{|\nu|}(\mu)$:

$$J_{|\nu|}(\mu_n) = J_{|\nu|}(\mu_m) = 0, \quad \mu_n > \mu_m > 0, \quad n > m.$$

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Note, that the definition in Eq.(27) automatically respects the vanishing boundary conditions for $u(\tau, z)$. We assume that this series converges absolutely and uniformly $\forall z \in [0, y(\tau)]$ for any $\tau > 0$.
Applying the direct integral transform in Eq.(23) to both parts of Eq.(27), and using a change of variables $z \to \hat{z} = zy(\tau)$ yields

$$\frac{\bar{u}(\tau, p)}{y^2(\tau)} = \sum_{n=1}^{\infty} \alpha_n(\tau) \int_0^1 \hat{z} J_{|\nu|}\left(\mu_n \hat{z}\right) J_{|\nu|}(py(\tau)\hat{z}) d\hat{z}. \tag{28}$$

The set of functions $J_{|\nu|}(\alpha\hat{z})$ with $\alpha \in \mu_n,\ n = 1, \ldots,$ forms an orthogonal basis in the space $C[0, 1]$ with the scalar product

$$\langle J_{|\nu|}(\alpha z), J_{|\nu|}(\beta z)\rangle = 2\int_0^1 \frac{z J_{|\nu|}(\alpha z) J_{|\nu|}(\beta z) dz}{J_{|\nu|+1}(\alpha) J_{|\nu|+1}(\beta)} = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta \end{cases} \tag{29}$$

Therefore, the explicit formula for each coefficient $\alpha_n(\tau)$ is straightforward

$$\alpha_n(\tau) = 2\frac{\bar{u}\left(\tau, \mu_n / y(\tau)\right)}{y^2(\tau) J^2_{|\nu|+1}(\mu_n)}. \tag{30}$$

Then the final solution for $u(\tau, z)$ reads

$$u(\tau, z) = 2\frac{z^{-\nu}}{y^2(\tau)} \sum_{n=1}^{\infty} \left[\int_0^{y(0)} u(0, s) s^{\nu+1} e^{-\frac{\mu_n^2}{2y^2(\tau)}\tau} \frac{J_{|\nu|}(\mu_n s/y(\tau)) J_{|\nu|}(\mu_n z/y(\tau))}{J^2_{|\nu|+1}(\mu_n)} ds \right. \tag{31}$$

$$\left. + \frac{1}{2} \int_0^{\tau} y^{\nu+1}(s) \Psi(s) e^{-\frac{\mu_n^2}{2y^2(\tau)}(\tau - s)} \frac{J_{|\nu|}(\mu_n y(s)/y(\tau)) J_{|\nu|}(\mu_n z/y(\tau))}{J^2_{|\nu|+1}(\mu_n)} ds \right].$$

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To obtain the equation for Ψ, one needs to differentiate Eq.(31) on z, and then let $z = y(\tau)$. This yields

$$\Psi(\tau) = -\frac{2}{y^{3+\nu}(\tau)} \sum_{n=1}^{\infty} (\mu_n + \nu) \left[\int_0^{y(0)} u(0, s) s^{\nu+1} e^{-\frac{\mu_n^2 \tau}{2y^2(\tau)}} \frac{J_{|\nu|}(\mu_n s/y(\tau))}{J_{|\nu|+1}(\mu_n)} ds \right. \tag{32}$$

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Note: The function $\Psi(\tau)$ can also be found by solving the following Fredholm equation:

$$\int_0^{\infty} e^{-\lambda^2 s/2}\Psi(s) y^{\nu+1}(s) I_\nu\left(y(s)\lambda\right) ds = -2\int_0^{y(0)} q^{\nu+1} I_\nu\left(q\lambda\right) u(0, q) dq. \tag{33}$$

The problem in half-plane domain $[y(\tau), \infty)$ can be solved via the Weber–Orr transform

$$\bar{u}(\tau, p) = \int_{y(\tau)}^{\infty} z^{\nu+1} W(\tau, p, z) u(\tau, z) dz \tag{34}$$

$$u(\tau, z) = z^{-\nu} \int_0^{\infty} \frac{pW(\tau, p, z)}{V(\tau, p)} \bar{u}(\tau, p) dp.$$

The kernel $W(a, b)$ and the function $V(p)$ are defined as follows, [Bateman and Erdélyi 1953]

$$W(\tau, a, b) = J_{|\nu|}(ab) Y_{|\nu|}(ay(\tau)) - Y_{|\nu|}(ab) J_{|\nu|}(ay(\tau)), \tag{35}$$

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The functions $W(\tau, a, b)$ as the functions of the second argument a also form an orthogonal basis in the space $C[y(\tau), \infty)$ for all $\tau > 0$. and the functions $J_\nu(xp)$ form an orthogonal basis in the space $C[0, \infty)$.

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$$u(\tau, z) = z^{-\nu} \int_0^{\infty} \frac{p W(\tau, p, z)}{V(\tau, p)} \bar{u}(\tau, p) dp.$$

The kernel $W(a, b)$ and the function $V(p)$ are defined as follows, [Bateman and Erdélyi 1953]

$$W(\tau, a, b) = J_{|\nu|}(ab) Y_{|\nu|}(a y(\tau)) - Y_{|\nu|}(ab) J_{|\nu|}(a y(\tau)), \tag{35}$$
$$V(\tau, p) = {J_{|\nu|}}^2(p y(\tau)) + {Y_{|\nu|}}^2(p y(\tau)).$$

The Webber–Orr transform is a generalization of the Hankel transform to the interval $[a, \infty)$.

The functions $W(\tau, a, b)$ as the functions of the second argument a also form an orthogonal basis in the space $C[y(\tau), \infty)$ for all $\tau > 0$. and the functions $J_\nu(xp)$ form an orthogonal basis in the space $C[0, \infty)$.

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In particular, for the indexes $\nu = 1/2 + k$, $k \in \mathbb{Z}$ the functions $J_{|\nu|}$ and $Y_{|\nu|}$ can be explicitly represented in terms of sine and cosine functions.

- We consider Up-and-Out Barrier Call option written on the underlying stock which follows the CEV process with $0 < \beta < 1$. To recall, after the change of variables proposed in this paper is done, the problem transforms to the solution of the Bessel PDE at the domain $\bar{z} \in [y(\tau), \infty)$. Hence, in new variables the Up-and-Out option transforms to the Down-and-Out option.
- We approach at pricing the Up-and-Out barrier Call option in the CEV model twofold. First, as a benchmark we solve the PDE in Eq.(2) by using a finite-difference (FD) scheme of the second order in space and time. We use the Crank-Nicolson scheme with few first Rannacher steps on a non-uniform grid compressed close to the barrier level, see [Itkin 2017] .

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In this test we use the explicit form of parameters $r(t), q(t), \sigma(t)$

$$r(t) = r_0 - r_k(a+t), \qquad q(t) = q_0 - q_k(a+t), \qquad \sigma(t) = \sigma_0\sqrt{a+t}, \tag{37}$$

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Table 1: Parameters of the test.

r_0	q_0	σ_0	r_k	q_k	σ_k	a	H	S
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We run the test for a set of maturities $T \in [1/12, 0.3, 0.5, 1]$ and strikes $K \in [59, 64, 69, 74, 79, 84]$.

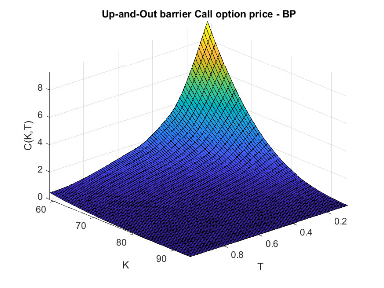

Figure 1: Up-and-Out barrier Call option price computed by using the BP method.

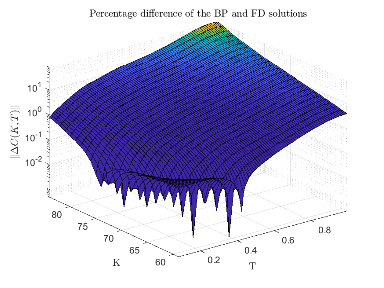

Figure 2: % difference of prices computed by using the BP and FD methods.

Table 2: Up-and-Out barrier Call option prices computed by using the BP and FD methods.

	BP				FD				Difference %			
K\T	**0.0833**	**0.3**	**0.5**	**1.0**	**0.0833**	**0.3**	**0.5**	**1.0**	**0.0833**	**0.3**	**0.5**	**1.0**
59	9.3192	3.3642	1.6845	0.4976	9.2924	3.3554	1.6884	0.5175	0.2876	0.2604	-0.2321	-3.9899
64	6.2167	2.1795	1.0671	0.3038	6.2025	2.1831	1.0793	0.3252	0.2286	-0.1654	-1.1438	-7.0291
69	3.8402	1.3219	0.6339	0.1731	3.8341	1.3319	0.6494	0.1931	0.1597	-0.7624	-2.4444	-11.5443
74	2.1608	0.7350	0.3450	0.0891	2.1605	0.7477	0.3606	0.1061	0.0118	-1.7293	-4.5240	-19.1029
79	1.0746	0.3612	0.1652	0.0388	1.0775	0.3736	0.1787	0.0522	-0.2700	-3.4102	-8.1779	-34.6241
84	0.4448	0.1462	0.0641	0.0121	0.4484	0.1561	0.0743	0.0216	-0.7971	-6.7649	-15.9277	-78.1360

- It can be seen that the agreement of both methods is good (less than 1%) if the option price is not too small which occurs when the strike K is close to the barrier or at high maturities. In this case, as this is seen from Table 2, the relative difference becomes large, but the absolute difference of two methods is about one cent, which is almost insignificant. Obviously, such cases are a challenge for any FD method, as at $t = T$ there is a jump in the initial condition at the boundary, and the first derivative of the solution doesn't exists in this point.
- As far as performance of both methods is concerned, to decrease the elapsed time for the FD method instead of Eq.(2) we solve the corresponding forward PDE. Therefore, prices of all options for a given set of strikes and maturities could be obtained within one sweep. This also requires $\bar{m} \times \bar{k}$ integrations of the product of thus found density function with the payoff function, where $\bar{m}$ is the total number of maturities, and $\bar{k}$ is the total number of strikes. In this test the elapsed time for the FD method is, on average, 140 msec.

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- For the BP method, since the expression for $\varsigma(\tau)$ in Eq.(17) is not known in closed form, we compute this integral numerically by using the Simpson quadratures. Nevertheless, to make the results accurate, we need to increase the number of the grid nodes M. As compared with [Itkin and Muravey 2020] , where a similar expression for the Hull-White model could be computed in closed form, and $M = 20$ provided a sufficient accuracy, here we need to take $M = 100$. Nevertheless, the elapsed time, on average, is 70 msec, i.e. twice faster than the FD method for the forward equation.

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Here we present the results of the perfomance of GIT method. Our numerical scheme is similar to that for the BP method: first we solve the Volterra equation and then compute the value of the integrals using a trapezoidal rule. The comparison with FD benchmark is presented in the following Figure and Table:

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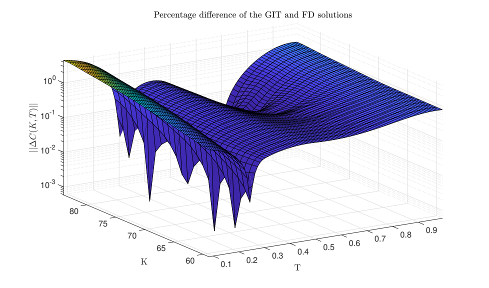

Figure 3: Percentage difference of Up-and-Out barrier Call option prices computed by using the FD and GIT methods.

	Difference %			
K\T	**0.0833**	**0.3**	**0.5**	**1.0**
59	0.71108	-0.2191	-0.3634	-0.7888
64	1.1101	-0.2149	-0.3654	-0.7997
69	1.7729	-0.20803	-0.3556	-0.8132
74	2.6313	-0.1879	-0.3318	-0.8992
79	3.5572	-0.1327	-0.3129	-1.0134
84	4.4718	0.0044	-0.2848	-1.1541

Table 3: Up-and-Out barrier Call option prices computed by using the GIT and FD methods.

- It can be seen that this method produces very accurate results at high strikes and maturities (i.e. where the option price is relatively small) in contrast to the BP method. The solution obtained by GIT consists of the exponents proportional to the time τ, see [Carr, Itkin, and Muravey 2020] . Contrary, when the price is higher (short maturities, low strikes) the GIT method is slightly less accurate than the BP method, as in Eq.(19) the exponent is inversely proportional to τ.
- This situation is well investigated for the heat equation with constant coefficients. As applied to pricing double barrier options, it is described in [Lipton 2002] . There exist two representation of the solution: one - obtained by using the method of images, and the other one - by the Fourier series. Despite both solutions are equal as the infinite series, their convergence properties are different. In particular, the Fourier method is superior when the difference between the upper H and lower L barriers is small and the time is relatively large. And the image expansion should be used otherwise.

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Thank you!