Looking Forward to Backward-Looking Rates:
Completing the Generalized Forward Market Model

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QuantMinds Americas
September 23rd, 2020

Introduction

- The generalized Forward Market Model (FMM) was introduced by L. and M. (2019) as a post-Libor extension of the classic Libor Market Model (LMM).
- The FMM accommodates both the traditional forward-looking (Libor-like) rates and the new setting-in-arrears backward-looking rates, which are expected to replace LIBORs in derivative contracts.
- Moreover, the FMM provides additional information about the rate dynamics between fixing/payment times.
- Like the LMM, however, the FMM is not a full term-structure model.
- In this talk, we show how to complete the FMM by inferring the evolution of generic term rates as well as of the short rate.
- This is crucial for the applicability of the model to the pricing and hedging of large heterogeneous portfolios.

Main assumptions, definitions and notation

The basic set up

- We consider a continuous-time financial market with an instantaneous risk-free rate, whose time-t value is denoted by $r(t)$.
- We assume that $r(t)$ is the collateral rate in standard CSAs, as well as the Price Alignment Interest for cleared derivatives.
- Rate $r(t)$ has an associated money-market account $B(t)$ such that

  $$dB(t) = r(t)B(t)\,dt$$

  and $B(0) = 1$, so $B(t) = e^{\int_0^t r(u)\,du}$.
- We assume the existence of a risk-neutral measure Q, whose associated numeraire is $B(t)$.
- We denote by $\mathbb{E}$ the expectation with respect to Q, and by $\mathcal{F}_t$ the "information" available in the market at time t.

- We then denote by $P(t,T)$ the price at time t of the extended zero-coupon bond with maturity T, that is:

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(u)\,\mathrm{d}u} | \mathcal{F}_t\right]$$

which is defined for any time t, even $t > T$.

- In fact, when $t > T$:

$$P(t,T) = \mathbb{E}\left[e^{\int_T^t r(u)\,\mathrm{d}u} | \mathcal{F}_t\right] = e^{\int_T^t r(u)\,\mathrm{d}u} = \frac{B(t)}{B(T)}$$

- The extended bond price $P(t,T)$ is the value of the self-financing strategy, and is therefore a viable numeraire.
- The martingale measure associated to the extended bond price $P(t,T)$ is called extended T-forward measure, and is denoted by Q^T.

- Given the time structure $0 = T_0, T_1, \ldots, T_M$, and denoting by τ_j the year fraction for the interval $[T_{j-1}, T_j)$, the daily-compounded setting-in-arrears rate for the interval $[T_{j-1}, T_j)$ is given by

$$R(T_{j-1}, T_j) = \frac{1}{\tau_j}\left[\prod_{i=1}^{n}(1 + r_i\delta_i) - 1\right]$$

  where the product is over the business days in $[T_{j-1}, T_j)$, and where r_i is the RFR fixing on date i with associated day-count fraction δ_i.

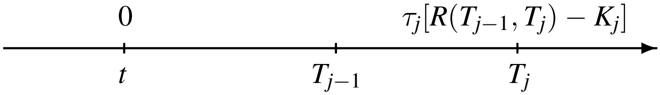
- For computational convenience, we approximate $R(T_{j-1}, T_j)$ as follows

$$R(T_{j-1}, T_j) \approx \frac{1}{\tau_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)\,\mathrm{d}u} - 1\right] = \frac{1}{\tau_j}\left[\frac{B(T_j)}{B(T_{j-1})} - 1\right]$$

Main assumptions, definitions and notation

Backward-looking in-arrears forward rates

- We define the backward-looking forward rate $R_j(t)$ at time t as the value of the fixed rate K_j in the swaplet paying $\tau_j[R(T_{j-1}, T_j) - K_j]$ at time T_j, such that the swaplet has zero value at time t:



- By no-arbitrage, for each time t, we have:

$$R_j(t) = \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j) | \mathcal{F}_t\right]$$

- A similar definition holds for the forward-looking forward rate $F_j(t)$.
- It is easy to show that $F_j(t) = R_j(t)$ for $t \leq T_{j-1}$, so process $R_j(t)$ can be used to model both rates at the same time.

Properties of the backward-looking forward rate $R_j(t)$

- A martingale under the T_j-forward measure:

$$R_j(t) = \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j)|\mathcal{F}_t\right]$$

- Can be written, for each generic time t, as

$$R_j(t) = \frac{1}{\tau_j}\left[\frac{P(t, T_{j-1})}{P(t, T_j)} - 1\right]$$

- Equal to the forward-looking spot rate at time T_{j-1}:

$$R_j(T_{j-1}) = F(T_{j-1}, T_j)$$

- Equal to the backward-looking spot rate at time T_j:

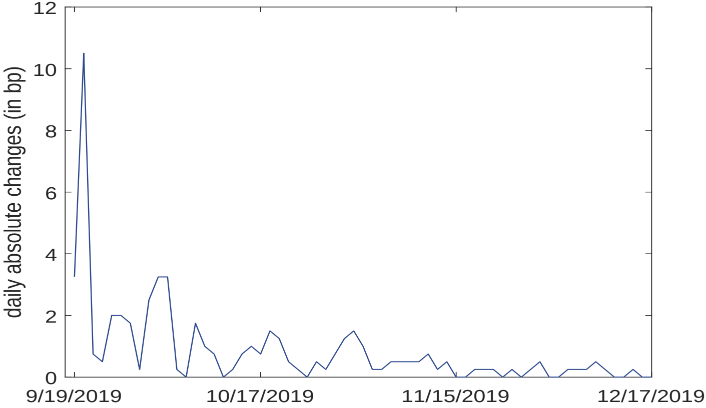
$$R_j(T_j) = R(T_{j-1}, T_j)$$

- Constant after time T_j:

$$R_j(t) = R(T_{j-1}, T_j), \quad t > T_j$$

The forward rate dynamics

- The dynamics of each $R_j(t)$ can be general for $t < T_{j-1}$, but its volatility must progressively decrease down to zero in $[T_{j-1}, T_j]$.
- This is also confirmed by SOFR futures data:



The forward rate dynamics

- Then, for each $j = 1, \ldots, M$, we assume that, under Q^{T_j},

$$\mathrm{d}R_j(t) = \sigma_j(t)\gamma_j(t)\,\mathrm{d}W_j(t)$$

  where $\sigma_j(t)$ is an adapted process, $W_j(t)$ is a standard Brownian motion, and $\gamma_j(t)$ is a deterministic function such that:
  - $\gamma_j(t) = 1$ for $t \leq T_{j-1}$
  - $\gamma_j(t)$ is differentiable and decreasing in (T_{j-1}, T_j)
  - $\gamma_j(t) = 0$ for $t \geq T_j$

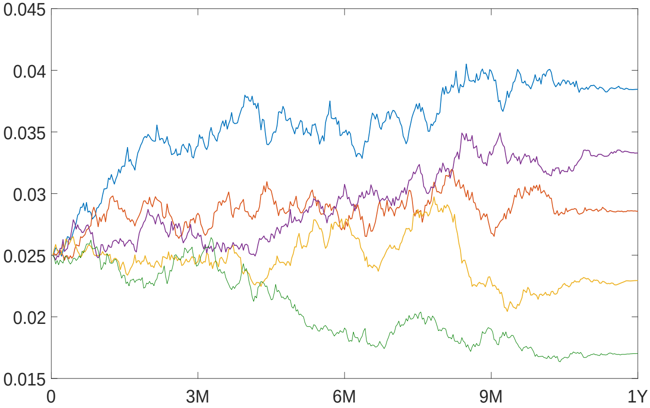
- In the Ho-Lee model, for instance, the function γ_j is piece-wise linear:

$$\gamma_j(t) = (T_j - t)/(T_j - T_{j-1}), \quad t \in (T_{j-1}, T_j)$$

- Contrary to the classic LMM case, the forward rate dynamics does not stop at time T_{j-1}, but R_j continues to evolve stochastically until T_j.

The forward rate dynamics

- Let us assume lognormal dynamics with volatility equal to 30%.
- Here is a plot of simulated paths of $R_j(t)$, where T_{j-1} =9M and T_j =1Y, for $t \in [0, T_j)$:



The generalized FMM

- The FMM is an extension of the classic single-curve LMM in that it models the joint dynamics not only of forward-looking forward rates, but also of backward-looking (setting-in-arrears) ones.
- Each rate R_j, $j = 1, \ldots, M$, has Q^{T_j}-dynamics given by:

$$\mathrm{d}R_j(t) = \sigma_j(t)\gamma_j(t)\,\mathrm{d}W_j(t)$$

  where we assume that $\mathrm{d}W_i(t)\,\mathrm{d}W_j(t) = \rho_{i,j}\,\mathrm{d}t$.

- Also in the FMM, we can specify the forward rates dynamics under:
  - The classic spot-Libor measure Q^d
  - A general T_k-forward measure Q^{T_k}
- In fact, the FMM dynamics under Q^d or Q^{T_k} are the same as those of the corresponding LMM.

- However, the big news is that the FMM allows for forward-rates dynamics under the risk-neutral money-market measure Q as well.
- In a classic LMM, the Q-dynamics of forward rates can be derived once we also model the volatility of the prompt zero-coupon bonds $P(t, T_{\eta(t)})$, where $\eta(t) = \min\{j : \ T_j \geq t\}$.
- In the FMM, the volatility of $P(t, T_{\eta(t)})$ is implicitly defined by:

$$1 + \tau_j R_j(t) = \frac{e^{\int_{T_{j-1}}^{t} r(s)\, \mathrm{d}s}}{P(t, T_j)}, \quad t \in [T_{j-1}, T_j]$$

- Therefore, we can derive the FMM dynamics under Q with no extra assumptions.
- Alternatively, we can recall that $P(t, 0) = B(t)$.

- The Q-dynamics of R_j can be written as:

$$\mathrm{d}R_j(t) = \sigma_j(t)\gamma_j(t) \sum_{i=\eta(t)}^{j} \rho_{i,j} \frac{\tau_i \sigma_i(t)\gamma_i(t)}{1+\tau_i R_i(t)}\,\mathrm{d}t + \sigma_j(t)\gamma_j(t)\,\mathrm{d}W_j^Q(t)$$

- Using vector notation, this becomes:

$$\mathrm{d}R_j(t) = \gamma_j(t)\sigma_j^R(t)^\intercal \sum_{i=\eta(t)}^{j} \sigma_i^R(t) \frac{\tau_i \gamma_i(t)}{1+\tau_i R_i(t)}\,\mathrm{d}t + \gamma_j(t)\sigma_j^R(t)^\intercal\,\mathrm{d}W(t)$$

  where $W(t)$ is an N-dimensional Brownian motion, and for each $j = 1, \dots, M$, $\sigma_j^R(t)$ is an N-dimensional adapted process: $\sigma_j^R(t)^\intercal = \sigma_j(t) C_j$, where C_j is the j-th row of C and $\rho = (\rho_{i,j})_{i,j=1,\dots,M} = CC^\intercal$.

- FMM rate dynamics under Q vs LMM rates dynamics under Q^d:

$$\mathrm{d}R_j(t) = \sigma_j(t)\gamma_j(t)\left[\sum_{i=\eta(t)}^{j}\rho_{i,j}\frac{\tau_i\sigma_i(t)\gamma_i(t)}{1+\tau_i R_i(t)}\,\mathrm{d}t + \mathrm{d}W_j^Q(t)\right]$$

$$\mathrm{d}L_j(t) = \sigma_j(t)\left[\sum_{i=\eta(t)+1}^{j}\rho_{i,j}\frac{\tau_i\sigma_i(t)}{1+\tau_i L_i(t)}\,\mathrm{d}t + \mathrm{d}W_j^Q(t)\right]$$

- When $\gamma_j(t) = 1_{\{t\leq T_{j-1}\}}$ (zero vol within each application period), then:
  - The FMM and LMM dynamics coincide
  - Discrete (spot-Libor) and continuous (money-market) risk-neutral measures coincide
  - Rate evolution is deterministic within each application period
  - Realized forward-looking and backward-looking rates are the same

The generalized FMM

- We can price contracts with fixings and payment times in the given grid $T_0, T_1, \ldots, T_M$, by simulating the FMM in the risk-neutral measure Q.
- But what about more complex contracts, such as range accruals or mortgages, which require simulation of off-grid rates or rates with different tenors?
- The problem of recovering the whole yield curve evolution in an LMM has been extensively studied in the financial literature, and is referred to as Libor-rate interpolation, or front- and back-stub interpolations.
- The main references are: Schlögl (2002), Piterbarg (2004), Beveridge and Joshi (2009), Werpachowski (2010), and Andersen and Piterbarg (2010).
- In the following, we propose an arbitrage-free method for generating off-grid, general-tenor rates and the bank account in an FMM.

FMM: the origin

- In the classic HJM framework, the instantaneous forward rate $f(t,T)$, for a fixed maturity T, evolves under Q according to:

$$\mathrm{d}f(t,T) = \sigma(t,T)^{\intercal} \int_t^T \sigma(t,s)\,\mathrm{d}s\,\mathrm{d}t + \sigma(t,T)^{\intercal}\,\mathrm{d}W(t)$$

  where $\sigma(t,T)$ is an N-dimensional vector of adapted processes, and W is an N-dimensional Q-Brownian motion.

- Volatility $\sigma(t,T)$, and hence process $f(t,T)$, are defined for $t \leq T$.

- To extend dynamics to times t after T, we zero the volatility of $f(t,T)$ for $t \geq T$, an idea already present in Rebonato (2002):

$$\mathrm{d}f(t,T) = 1_{\{t\leq T\}} \left[\sigma(t,T)^{\intercal} \int_t^T \sigma(t,s)\,\mathrm{d}s\,\mathrm{d}t + \sigma(t,T)^{\intercal}\,\mathrm{d}W(t)\right]$$

  so $f(t,T)$ is defined for all pairs (t,T) and $f(t,T) = r(T)$ when $t \geq T$.

FMM: the origin

- Application of Ito's lemma and Fubini's theorem leads to the following risk-neutral dynamics of extended zero-coupon bond prices:

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r(t)\,\mathrm{d}t - \left(\int_t^T \sigma(t,u) 1_{\{t\leq u\}}\,\mathrm{d}u\right)^{\mathsf{T}} \mathrm{d}W(t)$$

- In particular, when $t > T$, this SDE reduces to:

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r(t)\,\mathrm{d}t$$

  which is consistent with the definition of extended bond price after T.

- Using the above bond dynamics, we can derive the Q-dynamics of the generalized forward rate $R_j(t)$. We get:

$$\mathrm{d}R_j(t) = \cdots\,\mathrm{d}t + \left[R_j(t) + \frac{1}{\tau_j}\right] \left(\int_{T_{j-1}}^{T_j} \sigma(t,u) 1_{\{t\leq u\}}\,\mathrm{d}u\right)^{\mathsf{T}} \mathrm{d}W(t)$$

An extended Markovian HJM

- Our goal is to construct a Markovian HJM model that generates dynamics of forward rates R_j, $j = 1, \ldots, M$, that are equivalent to the FMM ones.
- Similarly to Cheyette (2001), we then assume that the instantaneous forward-rate volatility is given by the following separable form:

$$\sigma(t,T) = \sum_{k=1}^{M} \varsigma_k(t)\, g_k(T)\, 1_{\{T \in (T_{k-1}, T_k]\}}$$

  where, for each $k = 1, \ldots, M$, ς_k is an N-dimensional adapted process and g_k is a deterministic function.
- We set

$$G_k(t,T) = \int_t^T g_k(u)\, \mathrm{d}u$$

  and assume, with no loss of generality, that $G_k(T_{k-1}, T_k) = 1$.

An extended Markovian HJM

- Recalling that

$$\sigma(t,T) = \sum_{k=1}^{M} \varsigma_k(t)\, g_k(T)\, 1_{\{T\in(T_{k-1},T_k]\}}$$

the dynamics of $R_j(t)$ becomes:

$$\begin{aligned} \mathrm{d}R_j(t) &= \cdots\, \mathrm{d}t + \left[R_j(t) + \frac{1}{\tau_j}\right] \left(\int_{T_{j-1}}^{T_j} \sigma(t,u) 1_{\{t\le u\}}\, \mathrm{d}u\right)^{\mathsf{T}} \mathrm{d}W(t) \\ &= \cdots\, \mathrm{d}t + G_j(T_{j-1}\vee t, T_j \vee t) \left[R_j(t) + \frac{1}{\tau_j}\right] \varsigma_j(t)^{\mathsf{T}} \mathrm{d}W(t) \end{aligned}$$

- We finally compare this to

$$\mathrm{d}R_j(t) = \cdots\, \mathrm{d}t + \gamma_j(t)\sigma_j^R(t)^{\mathsf{T}}\, \mathrm{d}W(t)$$

and solve for $g_j(t)$ and $\varsigma_j(t)$.

An extended Markovian HJM

- To summarize, setting

$$\sigma(t,T) = \sum_{k=1}^{M} \varsigma_k(t)\, g_k(T)\, 1_{\{T\in(T_{k-1},T_k]\}}$$

with

$$g_k(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\gamma_k(t)\, 1_{\{t\in(T_{k-1},T_k)\}}$$

$$\varsigma_k(t) = \sigma_k^R(t)\frac{1}{R_k(t)+\frac{1}{\tau_k}}$$

leads to rates R_j that follow the postulated FMM dynamics.

- The resulting HJM model inherits the covariance structure of the FMM, because σ_j^R depends on the volatility σ_j and correlations $\rho_{i,j}$.
- In the classic LMM, $\sigma(t,T)$ is only defined for $t \leq T_{\eta(T)-1}$. In our generalized FMM, instead, $\sigma(t,T)$ is defined for $t \leq T$ since $\varsigma_k(t)$ is defined up to T_k.

An extended Markovian HJM

- As an example, we can consider the FMM generated, on the time grid $T_0, \ldots, T_M$, by a one-factor Hull-White (1990) model:

$$dr(t) = a[\theta(t) - r(t)]\,dt + \sigma\,dW(t)$$

  where a and σ are positive constants, and θ is a deterministic function.

- As is well known, this is equivalent to the following one-factor HJM model with volatility

$$\sigma(t, T) = \sigma\, e^{-a(T-t)}$$

- It is then easy to show that we can express this volatility in a separable form by setting

$$\varsigma_k(t) = \sigma \frac{e^{-a(T_{k-1}-t)} - e^{-a(T_k-t)}}{a}$$

$$g_k(T) = \frac{a}{e^{-a(T_{k-1}-T)} - e^{-a(T_k-T)}}\, 1_{\{T\in[T_{k-1},T_k]\}}$$

An extended Markovian HJM

- A Markovian representation for $f(t,T)$ is obtained by integrating its SDE:

$$f(t,T) = \begin{cases} f(0,T) + g(T)^{\intercal} X(t) + g(T)^{\intercal} Y(t) G(t,T) & \text{if } t < T \\ r(T) = f(0,T) + g(T)^{\intercal} X(T) & \text{if } t \geq T \end{cases}$$

where processes $X(t) = \{X_1(t), \ldots, X_M(t)\}^{\intercal}$ and $Y(t) = (Y_{k,h}(t))_{k,h=1,\ldots,M}$ are defined by

$$\begin{aligned} \mathrm{d}X(t) &= Y(t) g(t)\, \mathrm{d}t + \varsigma(t)\, \mathrm{d}W(t) \\ \mathrm{d}Y(t) &= \varsigma(t)\, \varsigma(t)^{\intercal}\, \mathrm{d}t \end{aligned}$$

with $X(0) = 0$ and $Y(0) = 0$, and where we denote by:

  - $g(T)$ the vector $\{g_1(T), \ldots, g_M(T)\}^{\intercal}$
  - $G(t,T)$ the vector $\{G_1(t,T), \ldots, G_M(t,T)\}^{\intercal}$
  - $\varsigma(t)$ the $M \times N$-matrix $(\varsigma_k(t)^{\intercal})_{k=1,\ldots,M}$

An extended Markovian HJM

- Zero-coupon bond prices are given, for $t < T$, by:

$$P(t,T) = P(0,t,T)\exp\left\{-G(t,T)^\top X(t) - \frac{1}{2}G(t,T)^\top Y(t)G(t,T)\right\}$$

  where $P(s,t,T) = \frac{P(s,T)}{P(s,t)}$, and by $P(t,T) = \exp\{-\int_t^T r(u)\,\mathrm{d}u\}$ for $t \geq T$.

- The bond price dynamics are Markovian in the factors $X_k(t)$, $Y_{k,h}(t)$, and $R_k(t)$, $k,h = 1,\ldots,M$ (plus any extra factor in $\sigma_k^R(t)$).
- There are at least $M(M+5)/2$ state variables to simulate.
- This is because bond prices depend on rates R_j only indirectly through $X(t)$ and $Y(t)$, so we end up repeating some calculations.
- A more efficient algorithm will be outlined next, where we show that the number of state variables only grows linearly.

Completing the curve using the FMM-fitted HJM

- We want to generate fixings of general forward-looking or backward-looking rates, using respectively the following formulas:

$$F(t,T) = \frac{1}{\tau(t,T)}\left[\frac{1}{P(t,T)} - 1\right]$$

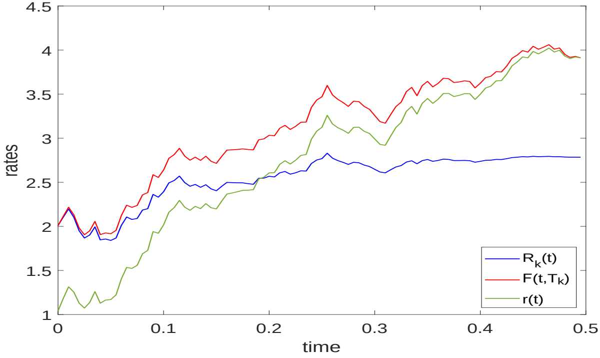
$$R(t,T) = \frac{1}{\tau(t,T)}\left[e^{\int_t^T r(u)\,\mathrm{d}u} - 1\right] = \frac{1}{\tau(t,T)}\left[\frac{B(T)}{B(t)} - 1\right]$$

  where $\tau(t,T)$ denotes the year fraction between t and T.

- Rates $F(t,T)$ can be calculated using the bond price formula above.
- Rates $R(t,T)$ can be obtained by integrating over paths of $r(t)$.
- These calculations, however, are typically too expensive to be implemented in a production code.
- We thus propose a much more efficient algorithm that leverages the value of simulated rates $R_j(t)$.

Completing the curve using the FMM-fitted HJM

- An example of evolution of rates $F(t, T_k)$, $R_k(t)$ and $r(t)$ within the k-th application period $[T_{k-1}, T_k]$ is shown in the figure below.
- We can see that: i) $R_k(t)$ exhibits volatility decay, ii) $R_k(t)$ and $F(t, T_k)$ coincide at $t = T_{k-1}$, and iii) $r(t)$ and $F(t, T_k)$ are driven by the same stochastic factor and converge to the same value at T_k.

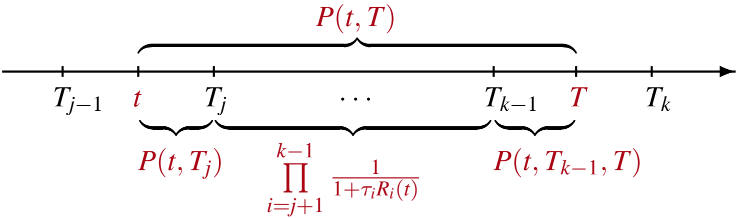


Completing the curve using the FMM-fitted HJM

- Following Schlögl (2002), for every (t, T) with $T > T_{\eta(t)}$, we can write:

$$P(t,T) = P(t, T_{\eta(t)}) \prod_{j=\eta(t)+1}^{\eta(T)-1} \frac{1}{1+\tau_j R_j(t)} \, P(t, T_{\eta(T)-1}, T)$$

- Visually:



- Discount factors $P(t, T_{\eta(t)})$ and $P(t, T_{\eta(T)-1}, T)$ are commonly referred to, respectively, as *front-stub* and *back-stub*.

Completing the curve using the FMM-fitted HJM

- Similarly to the LMM, the FMM can generate values of the central term of the previous formula

$$P(t,T) = P(t,T_{\eta(t)}) \prod_{j=\eta(t)+1}^{\eta(T)-1} \frac{1}{1+\tau_j R_j(t)} P(t,T_{\eta(T)-1},T)$$

  but not the discount factors $P(t,T_{\eta(t)})$ and $P(t,T_{\eta(T)-1},T)$.

- As per the bank account $B(t)$, we can write:

$$B(t) = P(t,T_{\eta(t)}) \prod_{j=1}^{\eta(t)} [1+\tau_j R_j(t)]$$

  which can be simulated by calculating the value of the front-stub $P(t,T_{\eta(t)})$.

The back-stub interpolation

- Lengthy but straightforward calculations lead to the following formula for the forward bond $P(t, T_{k-1}, T)$ with $t \le T_{k-1} < T \le T_k$:

$$P(t, T_{k-1}, T) = P(0, T_{k-1}, T)\big(1 + \tau_k R_k(t)\big)^{-G_k(T_{k-1},T)}$$
$$\cdot\, P(0, T_{k-1}, T_k)^{-G_k(T_{k-1},T)} \exp\left\{\frac{1}{2} G_k(T_{k-1}, T) G_k(T, T_k) Y_{k,k}(t)\right\}$$

- To simulate $P(t, T_{k-1}, T)$, for all possible t, T and k, we only need to simulate rates $R_k(t)$ and the diagonal elements of $Y(t)$ up to their fixing times:

$$Y_{k,k}(t) = \int_0^t \varsigma_k(s)^\intercal \varsigma_k(s)\,\mathrm{d}s = \int_0^t \left[\frac{\sigma_k(s)}{R_k(s) + \frac{1}{\tau_k}}\right]^2 \mathrm{d}s, \quad t \le T_{k-1}$$

- We also notice that $G_k(t, T) = \gamma_k(t) - \gamma_k(T)$, so:

$$G_k(T_{k-1}, T) = 1 - \gamma_k(T), \quad G_k(T, T_k) = \gamma_k(T)$$

The front-stub interpolation

- To derive the front-stub formula, we first derive a local version of the general bond-price formula when $\eta(t) = \eta(T) = k$:

$$P(t,T) = P(T_{k-1}, t, T)\exp\left\{-G_k(t,T)x_k(t) - \frac{1}{2}G_k^2(t,T)y_k(t)\right\}$$

where

$$dx_k(t) = g_k(t)y_k(t)\,dt + \frac{\sigma_k(t)}{R_k(t) + \frac{1}{\tau_k}}\,dW_k(t), \quad x_k(T_{k-1}) = 0$$

$$dy_k(t) = dY_{k,k}(t) = \left[\frac{\sigma_k(t)}{R_k(t) + \frac{1}{\tau_k}}\right]^2 dt, \quad y_k(T_{k-1}) = 0$$

- Therefore, $P(t,T)$, when $T_{k-1} < t < T \leq T_k$, can be simulated by simulating the local processes $x_k(t)$ and $y_k(t)$.
- The forward discount factor $P(T_{k-1}, t, T) = P(T_{k-1}, T)/P(T_{k-1}, t)$ can be calculated using the back-stub formula twice.

Summary of the simulation steps

- We assume that the FMM is simulated on a grid of times $0 = t_0, t_1, \ldots, t_m = T_M$, which contains the FMM dates $T_0, \ldots, T_M$.
- Starting from time-0 values, for $i = 1, \ldots, m$, we:
  - Simulate rates $R_k(t_i)$ and volatilities $\sigma_k(t_i)$ for each $k = \eta(t_i), \ldots, M$.
  - Calculate $Y_{k,k}(t_i)$ for each $k = \eta(t_i), \ldots, M$.
  - Calculate $y_k(t_i) = Y_{k,k}(t_i) - Y_{k,k}(T_{k-1})$ for each $k = \eta(t_i), \ldots, M$.
  - Simulate $x_k(t_i)$ for each $k = \eta(t_i), \ldots, M$.
  - Calculate $P(t_i, T)$ for all relevant times $t_i < T \leq T_{\eta(t_i)}$ using the front-stub formula.
  - Calculate $P(t_i, T)$ for all relevant times $T > T_{\eta(t_i)}$, where $P(t_i, T_{\eta(t_i)})$ is calculated using the front-stub formula, and $P(t_i, T_{\eta(T)-1}, T)$ using the back-stub formula.
  - Calculate the bank account $B(t_i)$, where $P(t_i, T_{\eta(t_i)})$ is calculated using the front-stub formula.
- We notice that each $Y_{k,k}(t)$ is simulated from time zero to time T_k, whereas each $x_k(t)$ evolves only in its period $[T_{k-1}, T_k]$.

Numerical examples

- We first tested our front-stub and back-stub formulas by numerically pricing caps on backward-looking 3M rates using Monte Carlo.
- We used a 6M-tenor shifted-lognormal FMM equivalent, on the given 6M-grid, to the Ho-Lee model with volatility parameter of 0.01, and a flat initial curve at 0%.
- In the following table, we report absolute differences in bp between the normal volatilities implied by corresponding numerical and analytical prices for a range of strikes and maturities.

Maturity/Strike	-2.0%	-1.5%	-1.0%	-0.5%	0.0%	0.5%	1.0%	1.5%	2.0%
1Y	0.441	0.201	0.146	0.214	0.169	0.211	0.148	0.185	0.424
2Y	0.158	0.216	0.292	0.325	0.313	0.332	0.303	0.245	0.210
3Y	0.012	0.149	0.302	0.418	0.460	0.440	0.340	0.209	0.089
5Y	0.179	0.293	0.410	0.507	0.543	0.544	0.486	0.405	0.342
10Y	0.045	0.108	0.171	0.221	0.257	0.261	0.229	0.197	0.176
15Y	0.008	0.046	0.089	0.127	0.151	0.155	0.138	0.122	0.112
20Y	0.063	0.014	0.037	0.082	0.114	0.129	0.128	0.125	0.124

Numerical examples

- Then, we considered an application to mortgages.
- We converted an existing LMM implementation into an FMM by: i) using our new back- and front-stub formulas; ii) flat-extrapolating LMM volatility parameters; iii) using a linear volatility decay.
- We looked at 30-year fixed-rate Uniform Mortgage Backed Securities that pay monthly cash flows, and priced their prepayment option using a 3M-tenor shifted-lognormal LMM/FMM.
- The following table reports absolute differences between FMM and LMM in the Option Adjusted Spread, Option Adjusted Duration and Weighted Average Life, for the most liquid coupons on 10/9/2019:

Coupon	OAS (basis points)	OAD (years)	WAL (years)
2.50	0.0500	0.0013	0.0036
3.00	0.0900	0.0011	0.0037
3.50	0.1100	0.0013	0.0035
4.00	0.1300	0.0017	0.0034
4.50	0.1200	0.0019	0.0027

Conclusions

- We showed that the generalized FMM introduced by L. and M. (2019) can be efficiently extended to make it a complete term-structure model.
- Generating off-grid rates along with the bank account is crucial to the pricing of general contracts and complex portfolios as well as to the valuation of Libor fallbacks.
- The FMM is completed by using a Markovian HJM that matches the FMM dynamics for the modeled forward rates, and which is used to derive back-stub and front-stub formulas to fill the FMM gaps.
- We also showed that the derived formulas are not only theoretically sound and arbitrage-free but also numerically efficient.
- Our FMM extension results in a model that is effectively a hybrid between an LMM and a Markovian HJM, and combines the flexibility of the former with the fine resolution of the latter, while preserving computational efficiency.

Disclaimer